

# ROWMOTION AND GENERALIZED TOGGLE GROUPS

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**ABSTRACT.** In this paper, we generalize the notion of the toggle group, as defined in [P. Cameron–D. Fon-der-Flaass '95] and further explored in [J. Striker–N. Williams '12], from the set of order ideals of a poset to any set  $\mathcal{L}$  of subsets of a countable set  $E$ . We prove a structure theorem for certain families of finite generalized toggle groups, similar to the theorem of Cameron and Fon-der-Flaass in the case of order ideals; we apply this theorem to several interesting settings, namely, chains, antichains, and interval-closed sets of a poset, as well as independent sets and vertex covers of a graph. We prove several other results in these toggle groups and also on matroids and convex geometries.

We further abstract the definition of generalized toggle groups to include subset-toggle groups and the power set toggle group; we show that the power set toggle group is the symmetric group.

Finally, if  $\mathcal{L}$  contains the closed sets of some closure operator, we generalize rowmotion on order ideals to an action we call cover-closure on  $\mathcal{L}$ . We prove that if cover-closure is bijective on some finite set of closed sets  $\mathcal{L}$ , then  $\mathcal{L}$  is a convex geometry; we also conjecture a stronger statement.

## 1. INTRODUCTION

In [4], P. Cameron and D. Fon-der-Flaass defined a permutation group on order ideals (or monotone Boolean functions) of a poset and proved that its group structure is, remarkably, the symmetric or alternating group whenever the Hasse diagram of the poset is connected. In [17], N. Williams and the author named this permutation group the *toggle group* and used the toggle group to prove results on the orbit structure of certain actions, finding easy proofs for instances of the *cyclic sieving phenomenon* of V. Reiner, D. Stanton, and D. White [13]. In [4], P. Cameron and D. Fon-der-Flaass also studied a certain convex-closure action on order ideals and proved it could be expressed as the toggle group action in which you compose all the toggles of the poset from top to bottom. In [17], N. Williams and the author named this action *rowmotion* and proved a theorem on the equivariance of rowmotion and *promotion* (loosely defined as toggling the elements of the poset from left to right). In [12], J. Propp and T. Roby began investigating a phenomenon they called *homomesy*. They built on the framework of [17] to prove instances of homomesy on the order ideals of the product of two chains poset under both rowmotion and promotion. Subsequently, D. Einstein and J. Propp [5] and D. Grinberg and T. Roby [8] extended the toggle group to the piecewise-linear and birational realms.

The papers cited above are part of the emerging area of *dynamical algebraic combinatorics* [1], defined as the study of dynamical systems arising from algebraic combinatorics. In this paper, we propose a research direction in this area by generalizing the notion of the toggle group **beyond the setting of order ideals**. We note that the particular structure of order ideals in a poset is unnecessary in the definition of the toggle group; the essential structure is that an order ideal is a subset of poset elements. Thus, given a countable ground set  $E$ , we define a (*generalized*) *toggle group*  $T(\mathcal{L})$  on **any set of subsets**  $\mathcal{L}$  of  $E$ ; see Definitions 2.1 and 2.3. Our philosophy is the following: if we isolate any set of subsets that has combinatorial meaning, we aim to use the toggle group to gain insight on these objects in ways similar to those outlined above for order ideals.

Our **first main theorem** is the following structure theorem for certain generalized toggle groups. (See Definitions 2.5, 2.10, and 2.12 for relevant definitions.)

**Theorem 2.13.** *Let  $E$  be a finite set, and  $\mathcal{L} \subseteq 2^E$ . If the toggle poset  $\mathcal{P}_{\mathcal{L}}$  is either a Cartesian product  $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} \times \mathcal{P}_{\mathcal{L}_2}$  or a toggle-disjoint direct sum  $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} + \mathcal{P}_{\mathcal{L}_2}$ , then  $T(\mathcal{L}) = T(\mathcal{L}_1) \times T(\mathcal{L}_2)$ .*

*Suppose that  $\mathcal{L}$  is a member of a pleasant family  $\mathcal{F}$ . Then,  $T(\mathcal{L})$  equals the symmetric group  $\mathfrak{S}_{\mathcal{L}}$  or the alternating group  $A_{\mathcal{L}}$ .*

The proof follows the proof of P. Cameron and D. Fon-der-Flaass [4] in the case of order ideals. So rather than introducing a new proof technique, Theorem 2.13 isolates the necessary qualities of the set of order ideals which produce this lovely toggle group description.

We apply this theorem to various combinatorially interesting sets of subsets to obtain the following corollary, which is actually a compilation of corollaries found in the listed sections.

**Corollary 2.14.** *Toggle groups of the following sets have group structure given by Theorem 2.13:*

- *Order ideals of a poset (proven in [4]),*
- *Chains of a poset (see Section 4.2),*
- *Antichains of a poset (see Section 4.3),*
- *Interval-closed sets of a certain family of posets (see Section 4.4),*
- *Independent sets of a graph (see Section 4.6), and*
- *Vertex covers of a graph (see Section 4.7).*

Notable results of Section 4, in addition to the corollaries in Corollary 2.14, include:

- Theorems 4.9 and 4.12, which prove the equivariance of some toggle orders in the chain and antichain toggle groups, respectively;
- Toggle commutation lemmas in all of the toggle groups listed in Corollary 2.14, as well as the acyclic subgraph toggle group;
- A characterization of the matroid independent set and basis exchange axioms in terms of toggles (Remarks 4.25 and 4.27);
- A characterization of convex geometries in terms of toggles (Remark 4.29);
- A definition of *extended convex geometries* (Definition 4.31);
- Proofs that the set of interval-closed sets of a poset is a convex geometry and that the set of chains of a poset is an extended convex geometry (Propositions 4.33 and 4.34).

As a further generalization of the toggle group, in Definitions 3.1, 3.3, and 3.5, we define subset-toggles, the power set toggle group, and subset-toggle groups. We show in Proposition 3.4 that the power set toggle group is always the symmetric group on  $\mathcal{L}$ .

In Section 5, we generalize rowmotion to an action we call *cover-closure*.

**Definition 5.2.** Let  $E$  be a countable set with closure operator  $\tau$ , and fix  $\mathcal{L} \subseteq 2^E$ . For  $X \in 2^E$ , let  $\text{cov}(X) \subseteq E \setminus X$  be the maximal subset of  $E \setminus X$  such that  $\forall e \in \text{cov}(X), X \cup \{e\} \in \mathcal{L}$ . Call  $\text{cov}(X)$  the set of *covers* of  $X$ . Then we define *cover-closure*  $\xi : 2^E \rightarrow 2^E$  as  $\xi(X) = \tau(\text{cov}(X))$ .

When restricted to order ideals of a poset, we show in Lemma 5.5 cover-closure is the usual rowmotion operation, defined in [4] and elsewhere as the order ideal generated by the minimal elements of the poset which are not already in the order ideal. (See [17] for further background on rowmotion.)

Our **second main theorem** is the following.

**Theorem 5.6.** *Given a finite ground set  $E$  and a closure operator  $\tau$  with set of closed sets  $\mathcal{L} \subseteq 2^E$ , if cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is injective and thus bijective, then  $\mathcal{L}$  must be a convex geometry.*

Finally, we conjecture the following strengthening of Theorem 5.6.

**Conjecture 5.8.** *Given a finite ground set  $E$  and a closure operator  $\tau$  with set of closed sets  $\mathcal{L} \subseteq 2^E$ , if cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is injective and thus bijective, then  $\mathcal{L}$  must be the set of order ideals  $J(P)$  for some poset  $P$ .*

The paper is structured as follows. Section 2 defines the generalized toggle group and proves Theorem 2.13: a structure description of the toggle group for certain families of subsets. Section 3 gives a further generalization to subset-toggle groups. Section 4 lists many examples of generalized toggle groups and gives some results in each case. Finally, Section 5 defines cover-closure on generalized toggle groups, relates it to rowmotion, and proves Theorem 5.6.

## 2. CONSTRUCTION AND STRUCTURE OF GENERALIZED TOGGLE GROUPS

For everything that follows, let  $E$  be a countable set and  $\mathcal{L}$  be any subset of the power set  $2^E$ .

**Definition 2.1.** For each element  $e \in E$  define its *toggle*  $t_e : \mathcal{L} \rightarrow \mathcal{L}$  as follows.

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in \mathcal{L} \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in \mathcal{L} \\ X & \text{otherwise} \end{cases}$$

We call  $\{t_e \mid e \in E\}$  the set of *(single-)toggles*.

*Remark 2.2.* Note that  $t_e^2 = 1$  for all  $e \in E$ .

We define the generalized (single-)toggle group as the group generated by these toggles.

**Definition 2.3.** Let  $T(\mathcal{L})$  be the subgroup of the symmetric group  $\mathfrak{S}_{\mathcal{L}}$ , generated by  $\{t_e \mid e \in E\}$ . Call  $T(\mathcal{L})$  the *(single-)toggle group* on  $\mathcal{L}$ .

**Example 2.4.** Let  $E = \{1, 2, 3, 4\}$  and  $\mathcal{L} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ .

Then, for example,

$$t_4(\{1, 2, 3\}) = \{1, 2, 3, 4\}, \quad t_2(\{1, 2\}) = \{1\}, \quad t_2(\{1, 2, 3, 4\}) = \{1, 2, 3, 4\}.$$

We can write  $t_1, t_2, t_3, t_4$  as permutations in cycle notation as follows.

$$\begin{aligned} t_1 &= (\emptyset, \{1\}) \\ t_2 &= (\{1\}, \{1, 2\})(\{1, 3\}, \{1, 2, 3\}) \\ t_3 &= (\{1\}, \{1, 3\})(\{1, 2\}, \{1, 2, 3\}) \\ t_4 &= (\{1, 2, 3\}, \{1, 2, 3, 4\}) \end{aligned}$$

We now define the *toggle poset*, which we will use in Theorem 2.13.

**Definition 2.5.** Let  $E$  be a finite set, and  $\mathcal{L} \subseteq 2^E$ . The *toggle poset*  $\mathcal{P}_{\mathcal{L}}$  of  $\mathcal{L}$  is defined as the partial order on  $\mathcal{L}$  where the covering relations are given by  $X \lessdot Y$  if and only if  $X \subseteq Y$  and  $|Y \setminus X| = 1$ , that is, if there exists  $e \in E$  such that  $t_e(X) = Y$ .

See Figures 1 and 2 for examples.

In the following proposition, we give some properties (and non-properties) of the toggle poset.

**Proposition 2.6.**  $\mathcal{P}_{\mathcal{L}}$  has the following properties:

- (1) The Hasse diagram of  $\mathcal{P}_{\mathcal{L}}$  need not be connected.
- (2)  $\mathcal{P}_{\mathcal{L}}$  is graded but not necessarily strongly graded.
- (3) If you label the edges of the Hasse diagram of  $\mathcal{P}_{\mathcal{L}}$  by the appropriate toggle, each toggle may appear at most once in any saturated chain.
- (4)  $\mathcal{P}_{\mathcal{L}}$  is not always the same as partially ordering  $\mathcal{L}$  by containment.

*Proof.* For part (2), there is a rank function  $\rho : \mathcal{P}_{\mathcal{L}} \rightarrow \mathbb{Z}$  given by  $\rho(X) = |X|$  and covering relations only appear between elements on adjacent ranks. Thus  $\mathcal{P}_{\mathcal{L}}$  is graded. To show  $\mathcal{P}_{\mathcal{L}}$  might not be strongly graded, consider as an example  $E = \{1, 2, 3\}$  and  $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{1, 3\}\}$ . Both  $\{\emptyset, \{2\}\}$

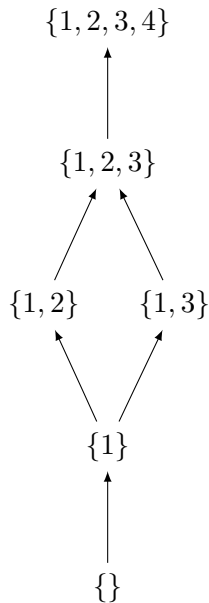


FIGURE 1. The toggle posets  $\mathcal{P}_{\mathcal{L}}$ , where  $\mathcal{L}$  is as in Example 2.4

and  $\{\emptyset, \{1\}, \{1, 3\}\}$  are maximal chains, but they do not have the same length. Since maximal chains need not have the same length,  $\mathcal{P}_{\mathcal{L}}$  is not strongly graded.

For part (3), note that each cover adds a single element to the subset and removes no elements. No element may be added twice, so each Hasse diagram edge in a saturated chain must represent a different toggle.

For part (4), consider as an example  $E = \{1, 2, 3\}$  and  $\mathcal{L} = \{\{1\}, \{1, 2, 3\}\}$ . In  $\mathcal{P}_{\mathcal{L}}$ ,  $\{1\}$  and  $\{1, 2, 3\}$  are incomparable, since there is no single toggle that maps  $\{1\}$  to  $\{1, 2, 3\}$ . This example also shows part (1).  $\square$

**Example 2.7.** In the case where  $\mathcal{L}$  equals the set of order ideals of a poset  $P$ , the toggle poset  $\mathcal{P}_{\mathcal{L}}$  is the distributive lattice of order ideals  $J(P)$ . Thus, in contrast to the general case of Proposition 2.6 parts (1), (2), and (4) discussed above,  $\mathcal{P}_{\mathcal{L}} = J(P)$  has connected Hasse diagram, is strongly graded, and is equivalent to the partial order by containment.

In the order ideal case  $\mathcal{L} = J(P)$ , P. Cameron and D. Fon-der-Flaass proved a structure theorem for  $T(\mathcal{L})$  [4].

**Theorem 2.8** ([4]). *Let  $P$  be a poset and  $J(P)$  the set of order ideals of  $P$ . If  $P$  is not the disjoint union of two posets,  $T(J(P))$  is either the symmetric group  $\mathfrak{S}_{J(P)}$  or the alternating group  $A_{J(P)}$ . If  $P$  is the disjoint union of two posets  $P = P_1 + P_2$ , then  $T(J(P)) = T(J(P_1)) \times T(J(P_2))$ .*

Though initial computations (including all toggle groups with  $|E| \leq 3$  and many other larger examples) indicated that all generalized toggle groups may be a direct product of symmetric and/or alternating groups as in the case of order ideals, this is not true, as shown by the following example.

**Example 2.9.** Let  $\mathcal{L} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{2, 3, 4\}, \{3, 4\}, \{4\}\}$ . We have computed using Sage [14] that the toggle group  $T(\mathcal{L})$  as a subgroup of  $\mathfrak{S}_{\mathcal{L}} \cong \mathfrak{S}_8$  (via the isomorphism in which the  $i$ th element of  $\mathcal{L}$  is mapped to  $i$ ) is given by the presentation

$$\langle (1\ 4)(3\ 7), (2\ 7)(4\ 5), (2\ 8)(5\ 6), (1\ 8)(3\ 6) \rangle.$$

We used this presentation to compute, using GAP [7] (via Sage [14]), that  $T(\mathcal{L}) = (((C_2 \times D_8) \rtimes C_2) \rtimes C_3) \rtimes C_2$  where  $C_n$  is the cyclic group of order  $n$  and  $D_8$  is the dihedral group of order 8. See Figure 2 for the toggle poset of this example.

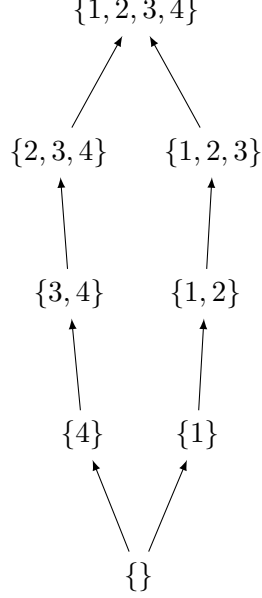


FIGURE 2. The toggle poset  $\mathcal{P}_{\mathcal{L}}$ , where  $\mathcal{L}$  is as in Example 2.9

Even though, by the above example, we see that generalized toggle group structure is not well-behaved in general, we show in this paper that similar structure theorems to Theorem 2.8 do exist in many other *combinatorially significant* generalized toggle groups. Our first main result, Theorem 2.13, gives sufficient conditions for  $T(\mathcal{L})$  to have a structure theorem similar to Theorem 2.8. We then summarize in Corollary 2.14 (and prove in Section 4) that the following cases satisfy the sufficient condition of Theorem 2.13: chains, antichains, and interval-closed sets of a poset; independent sets and vertex covers of a graph.

We will need the following definitions.

**Definition 2.10.**  $\mathcal{P}_{\mathcal{L}}$  is a *toggle-disjoint direct sum* ( $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} + \mathcal{P}_{\mathcal{L}_2}$ ) if  $\mathcal{P}_{\mathcal{L}}$  is a disjoint union of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  (that is, the Hasse diagram of  $\mathcal{P}_{\mathcal{L}}$  is disconnected) such that the toggles  $t_{e_1}$  corresponding to the Hasse diagram edges of  $\mathcal{P}_{\mathcal{L}_1}$  are such that  $e_1 \in E_1 \subset E$  and the toggles  $t_{e_2}$  corresponding to the Hasse diagram edges of  $\mathcal{P}_{\mathcal{L}_2}$  are such that  $e_2 \in E_2 \subset E$ , and  $E_1 \cap E_2 = \emptyset$ .

**Definition 2.11.** Given  $e \in E$  and  $\mathcal{L} \subseteq 2^E$ , define  $\mathcal{L}_e := \{X \in \mathcal{L} \mid e \in X\}$  to consist of all the subsets in  $\mathcal{L}$  that contain  $e$  and  $\mathcal{L}_{\bar{e}} := \{X \in \mathcal{L} \mid e \notin X\}$  to be all the subsets in  $\mathcal{L}$  that do not contain  $e$ .

**Definition 2.12.** Let  $\mathcal{F}$  be a family of sets of subsets of some ground set  $E$ . Call  $\mathcal{F}$  *pleasant* if each  $\mathcal{L} \in \mathcal{F}$  satisfies the properties below.

- (1)  $\mathcal{P}_{\mathcal{L}}$  is neither a Cartesian product nor a toggle-disjoint direct sum.
- (2) If there exists  $E' \subseteq E$  such that  $|E'| \leq 5$  and  $\mathcal{L} \subseteq 2^{E'}$ , then  $T_{\mathcal{L}} = \mathfrak{S}_{\mathcal{L}}$  or  $A_{\mathcal{L}}$ .
- (3) If there does not exist  $E' \subseteq E$  such that  $|E'| \leq 5$  and  $\mathcal{L} \subseteq 2^{E'}$ , then there exists  $e \in E$  so that  $\mathcal{L}_e \neq \emptyset$ ,  $\mathcal{L}_e \neq \mathcal{L}$ , and at least one of the following holds:
  - (a)  $\mathcal{L}_e$  is in  $\mathcal{F}$ ,  $\mathcal{L}_e \cup t_e(\mathcal{L}_e) = \mathcal{L}$ , and  $\mathcal{L}_e \cap t_e(\mathcal{L}_e) \neq \emptyset$ , or

(b)  $\mathcal{L}_{\bar{e}}$  is in  $\mathcal{F}$ ,  $\mathcal{L}_{\bar{e}} \cup t_e(\mathcal{L}_{\bar{e}}) = \mathcal{L}$ , and  $\mathcal{L}_{\bar{e}} \cap t_e(\mathcal{L}_{\bar{e}}) \neq \emptyset$ .

**Theorem 2.13.** *Let  $E$  be a finite set, and  $\mathcal{L} \subseteq 2^E$ . If  $\mathcal{P}_{\mathcal{L}}$  is either a Cartesian product  $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} \times \mathcal{P}_{\mathcal{L}_2}$  or a toggle-disjoint direct sum  $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} + \mathcal{P}_{\mathcal{L}_2}$ , then  $T(\mathcal{L}) = T(\mathcal{L}_1) \times T(\mathcal{L}_2)$ .*

*Suppose that  $\mathcal{L}$  is a member of a pleasant family  $\mathcal{F}$ . Then,  $T(\mathcal{L}) = \mathfrak{S}_{\mathcal{L}}$  or  $A_{\mathcal{L}}$ .*

*Proof.* Suppose  $\mathcal{P}_{\mathcal{L}}$  is a toggle-disjoint direct sum  $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} + \mathcal{P}_{\mathcal{L}_2}$ . So there exist  $E_1, E_2 \subset E$  with  $E_1 \cap E_2 = \emptyset$  such that the toggles  $t_{e_1}$  corresponding to the Hasse diagram edges of  $\mathcal{P}_{\mathcal{L}_1}$  are such that  $e_1 \in E_1$  and the toggles  $t_{e_2}$  corresponding to the Hasse diagram edges of  $\mathcal{P}_{\mathcal{L}_2}$  are such that  $e_2 \in E_2$ . Then  $t_{e_1}t_{e_2} = t_{e_2}t_{e_1}$  for all  $e_1 \in E_1$  and  $e_2 \in E_2$  since toggles in  $E_1$  affect no elements in  $\mathcal{L}_2$  and vice versa. Thus  $T(\mathcal{L}) = T(\mathcal{L}_1) \times T(\mathcal{L}_2)$ .

Suppose  $\mathcal{P}_{\mathcal{L}}$  is a Cartesian product  $\mathcal{P}_{\mathcal{L}} = \mathcal{P}_{\mathcal{L}_1} \times \mathcal{P}_{\mathcal{L}_2}$ . Then we can represent the elements of  $\mathcal{P}_{\mathcal{L}}$  as ordered pairs  $(X_1, X_2)$  where  $X_1 \in \mathcal{P}_{\mathcal{L}_1}, X_1 \subseteq E_1$  and  $X_2 \in \mathcal{P}_{\mathcal{L}_2}, X_2 \subseteq E_2$ ,  $E_1, E_2 \subset E$  with  $E_1 \cap E_2 = \emptyset$ , and the partial order is given by  $(X_1, X_2) \leq (Y_1, Y_2)$  if and only if  $X_1 \leq Y_1$  in  $\mathcal{P}_{\mathcal{L}_1}$  and  $X_2 \leq Y_2$  in  $\mathcal{P}_{\mathcal{L}_2}$ . The toggle groups  $T(\mathcal{L}_1)$  and  $T(\mathcal{L}_2)$  act on their respective components, so we have  $T(\mathcal{L}) = T(\mathcal{L}_1) \times T(\mathcal{L}_2)$ .

Now suppose  $\mathcal{L}$  is a member of a pleasant family  $\mathcal{F}$ , so by condition (1) of Definition 2.12,  $\mathcal{P}_{\mathcal{L}}$  is neither a Cartesian product or a toggle-disjoint direct sum. The proof follows the proof of Cameron and Fon-der-Flaass [4] for the order ideal toggle group and is by induction on the size of the ground set of  $\mathcal{L}$ . Let  $e$  be the element required by  $\mathcal{F}$  being a pleasant family. So we know  $\mathcal{L}$  is the disjoint union of  $\mathcal{L}_e$  and  $\mathcal{L}_{\bar{e}}$ , and neither  $\mathcal{L}_e$  nor  $\mathcal{L}_{\bar{e}}$  are empty. Suppose  $\mathcal{F}$  satisfies condition (3a) of Definition 2.12, so that  $|\mathcal{L}_e| \geq |\mathcal{L}_{\bar{e}}|$  (if not, then  $\mathcal{F}$  satisfies condition (3b), so  $|\mathcal{L}_e| \leq |\mathcal{L}_{\bar{e}}|$ , and the argument follows symmetrically with  $\mathcal{L}_{\bar{e}}$  instead). Now since all the elements of  $\mathcal{L}_e$  contain  $e$ ,  $t_e$  will act as the identity when regarded as a generator of  $T(\mathcal{L}_e)$ . So we may consider the toggle group  $T(\mathcal{L}_e)$  to be over ground set  $E \setminus \{e\}$ . By induction,  $T(\mathcal{L}_e)$  is at least the alternating group  $A_{\mathcal{L}_e}$ . Assume  $|\mathcal{L}_e| \geq 5$  so that  $A_{\mathcal{L}_e}$  is simple. Since  $A_{\mathcal{L}_e}$  is simple and  $|\mathcal{L}_e| > |\mathcal{L}_{\bar{e}}|$ , the subgroup  $K$  of  $T(\mathcal{L})$  fixing  $\mathcal{L}_{\bar{e}}$  pointwise induces at least  $A_{\mathcal{L}_e}$  on  $\mathcal{L}_e$ . (In the case  $|\mathcal{L}_e| \leq 5$ , (2) in Definition 2.12 gives that the subgroup  $K$  of  $T(\mathcal{L})$  fixing  $\mathcal{L}_{\bar{e}}$  pointwise induces at least  $A_{\mathcal{L}_e}$  on  $\mathcal{L}_e$ .) For any subset  $S \subseteq \mathcal{L}$ , let  $t_e(S) = \{t_e(s) \mid s \in S\}$ . Now  $\mathcal{L}_e \cup t_e(\mathcal{L}_{\bar{e}}) = \mathcal{L}$  and  $\mathcal{L}_e \cap t_e(\mathcal{L}_{\bar{e}}) \neq \emptyset$ , so  $K$  and the conjugate subgroup  $K^{t_e} (= \{t_e k t_e \mid k \in K\})$  generate an alternating or symmetric group on  $\mathcal{L}$ .  $\square$

In the following corollary, we apply Theorem 2.13 to obtain a toggle group structure description for various combinatorially interesting sets of subsets. We prove this corollary in each case in Section 4 by the following method. We first show when  $\mathcal{P}_{\mathcal{L}}$  is a Cartesian product or a toggle-disjoint direct sum and use the above theorem to say  $T(\mathcal{L})$  is a direct product in those cases. Secondly, in the case that  $\mathcal{P}_{\mathcal{L}}$  is not a Cartesian product or a toggle-disjoint direct sum, we show that  $\mathcal{L}$  is a member of a pleasant family, that is, there exists an  $e$  satisfying condition (3) in Definition 2.12. Finally, we must prove the base cases (condition (2) in Definition 2.12), namely, for  $|\mathcal{L}| \leq 5$ ,  $T(\mathcal{L}) = \mathfrak{S}_{\mathcal{L}}$  or  $A_{\mathcal{L}}$ .

**Corollary 2.14.** *Toggle groups of the following sets have group structure given by Theorem 2.13:*

- *Order ideals of a poset (proven in [4], see Section 4.1),*
- *Chains of a poset (see Section 4.2),*
- *Antichains of a poset (see Section 4.3),*
- *Interval-closed sets of a certain family of posets (see Section 4.4),*
- *Independent sets of a graph (see Section 4.6), and*
- *Vertex covers of a graph (see Section 4.7).*

In Section 4, we discuss these toggle groups in greater detail and prove some additional results. We also consider several other examples, in addition to the above list, namely, more than one partial order on the same ground set (Section 4.5), acyclic subgraphs of a graph (Section 4.8),

connected subgraphs of a graph (Section 4.9), matroids (Section 4.10), and convex geometries (or antimatroids) (Section 4.11).

Before stating and proving these results, we define a further extension of the toggle group to the *subset-toggle group*. The reader interested mainly in Theorem 2.13 and Corollary 2.14 may skip directly to Section 4.

### 3. CONSTRUCTION OF THE POWER SET TOGGLE GROUP AND SUBSET-TOGGLE GROUPS

We now construct an even more general toggle group. Rather than defining toggles for single elements in  $E$ , we extend this notion to the toggle  $t_S$  of a subset  $S \subseteq E$ .

As before, let  $E$  be a countable set and  $\mathcal{L}$  be any subset of the power set  $2^E$ .

**Definition 3.1.** For any subset  $S \subseteq E$  define its *(subset-)toggle*  $t_S : \mathcal{L} \rightarrow \mathcal{L}$  as

$$t_S(X) = \begin{cases} X \triangle S & \text{if } X \triangle S \in \mathcal{L} \\ X & \text{otherwise} \end{cases}$$

where  $X \triangle S$  denotes the symmetric difference of the sets  $X$  and  $S$ , that is,  $X \triangle S = (X \setminus S) \cup (S \setminus X)$ . We call  $\{t_S \mid S \subseteq E\}$  the set of *(subset-)toggles*.

*Remark 3.2.* Note that the single-toggles of Definition 2.1 are subset-toggles for which the size of the subset is one. Also note that the subset-toggle  $t_S$  for  $|S| > 1$  is not, in general, the product of the single-toggles  $t_{e_1} t_{e_2} \cdots t_{e_j}$  for  $S = \{e_1, \dots, e_j\}$ . In particular,  $t_{e_1} t_{e_2} \cdots t_{e_j}(X)$  could be something other than  $X \triangle S$  or  $X$  if some of the  $t_{e_i}$  act as the identity while others do not. The subset-toggle  $t_S$  toggles the *entire subset*  $S$  all at once (rather than sequentially), if possible, and otherwise acts as the identity.

We define the power set toggle group as the group generated by all the (subset-)toggles on  $\mathcal{L}$ .

**Definition 3.3.** Let  $T_{2^E}(\mathcal{L})$  be the subgroup of the symmetric group  $\mathfrak{S}_{\mathcal{L}}$ , generated by  $\{t_S \mid S \subseteq E\}$ . Call  $T_{2^E}(\mathcal{L})$  the *power set toggle group* on  $\mathcal{L}$ .

The power set toggle group is easy to characterize.

**Proposition 3.4.** The power set toggle group  $T_{2^E}(\mathcal{L})$  equals the symmetric group  $\mathfrak{S}_{\mathcal{L}}$ .

*Proof.* To show  $T_{2^E}(\mathcal{L}) = \mathfrak{S}_{\mathcal{L}}$ , we need only show that there exists a subset toggle to take any element of  $\mathcal{L}$  to any other. So suppose  $X, Y \in \mathcal{L}$ . Then we have  $t_{X \triangle Y}(X) = Y$ .  $\square$

One could also construct a toggle group using only some of the subset-toggles. Such toggle groups are always subgroups of the power set toggle group, since they are defined using a subset of the power set toggle group generators.

**Definition 3.5.** Let  $\mathcal{K} \subseteq 2^E$ . Define  $T_{\mathcal{K}}(\mathcal{L})$  to be the subgroup of  $T_{2^E}(\mathcal{L})$  generated by  $\{t_S \mid S \in \mathcal{K}\}$ . Call  $T_{\mathcal{K}}(\mathcal{L})$  the  $\mathcal{K}$ -toggle subgroup on  $\mathcal{L}$  (or, generically, we call any  $T_{\mathcal{K}}(\mathcal{L})$  a *subset-toggle group*).

We define a graph corresponding to each subset-toggle group.

**Definition 3.6.** Let  $E$  be a finite set, and  $\mathcal{K}, \mathcal{L} \subseteq 2^E$ . The *subset-toggle graph*  $G_{\mathcal{K}}(\mathcal{L})$  of  $T_{\mathcal{K}}(\mathcal{L})$  is defined as the graph with vertices  $\mathcal{L}$  and an edge from  $X$  to  $Y$  whenever there exists a subset  $S \in \mathcal{K}$  such that  $t_S(X) = Y$ .

Note we define a toggle graph rather than the toggle poset of Definition 2.5; unlike in the case of the single-toggle group, subset-toggles do not act by simply adding or removing one element. In fact, a subset-toggle may act nontrivially and leave the cardinality of the subset unchanged, as seen in the following example. Thus, you cannot always define a poset by using the subset-toggles as covers.

**Example 3.7.** Let  $E = \{1, 2, 3\}$ ,  $\mathcal{K} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ , and  $\mathcal{L} = \{\{1\}, \{2\}, \{3\}\}$ . Then, for example,  $t_{\{1,2\}}(\{1\}) = \{2\}$ . The toggle graph  $G_{\mathcal{K}}(\mathcal{L})$  in this case is given in Figure 3.

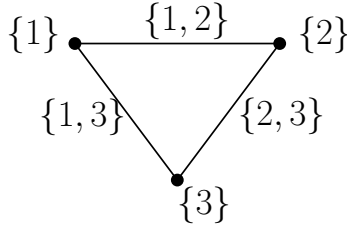


FIGURE 3. The subset-toggle graph  $G_{\mathcal{K}}(\mathcal{L})$  where  $\mathcal{K}$  and  $\mathcal{L}$  are as in Example 3.7 and the edges are labeled by the subset-toggle which maps one vertex to the other

In the next section, we begin to study various specific toggle groups of combinatorial significance.

#### 4. EXAMPLES

In this section, we look at many examples of toggle groups on combinatorial objects which are given as subsets of some finite ground set. For many of the examples, we prove a lemma concerning when toggles commute; in some cases, we give a structure theorem. The toggle group on order ideals  $J(P)$  of a poset  $P$ , as studied in [4] and [17], serves as the motivating example; we list some relevant properties and theorems in Section 4.1 for the sake of comparison.

This is neither an exhaustive list nor a complete treatment of any of these examples. Rather, we aim to identify these generalized toggle groups as excellent avenues for further study and initiate the investigation in each case. Unless otherwise noted, we restrict our attention to the single-toggle group, though subset-toggle groups would also be of interest.

Sections 4.1–4.5 study various poset toggle groups. (See Chapter 3 of [15] for background on posets.) Sections 4.6–4.9 examine toggle groups on graphs. Section 4.10 considers matroids and Section 4.11 discusses convex geometries (equivalent to antimatroids or meet-distributive lattices).

**4.1. Order ideals.** Given a poset  $P$ , an *order ideal*  $I$  is a subset  $I \subseteq P$  such that if  $y \in I$  and  $x \leq y$  then  $x \in I$ . Denote as  $J(P)$  the set of order ideals of  $P$ . Since order ideals are subsets of the elements of  $P$ , we take  $P$  as our ground set and  $\mathcal{L} = J(P) \subseteq 2^P$ .

In this subsection, we list, for the sake of comparison, the toggle commutation lemma and toggle group structure theorem of P. Cameron and D. Fon-der-Flaass [4] for the order ideal toggle group  $T(J(P))$ . We give proofs as motivating examples for subsequent subsections.

**Lemma 4.1** ([4]). *In the order ideal toggle group  $T(J(P))$ ,  $(t_p t_q)^2 = 1$  if and only if there is no covering relation between  $p$  and  $q$ .*

*Proof.* Suppose  $q$  covers  $p$ . Let  $I$  be the minimal order ideal containing every poset element that  $q$  covers. Then  $t_p t_q(I) = I \cup \{q\}$  whereas  $t_q t_p(I) = I \setminus \{p\}$ . So  $t_p$  and  $t_q$  do not commute.

If  $p < q$  but there exists  $r$  such that  $p < r < q$ , then for any  $I \in J(P)$ , if  $r \in I$ ,  $t_p(I) = I$ , and if  $r \notin I$ , then  $t_q(I) = I$ . So  $t_p$  and  $t_q$  commute.

If  $p$  and  $q$  are incomparable, then the presence or absence of  $p$  in  $I$  does not affect whether  $q$  may be in  $I$ . So  $t_p$  and  $t_q$  commute.  $\square$

The following is the theorem of P. Cameron and D. Fon-der-Flaass [4] on the order ideal toggle group structure. We give a proof below using Theorem 2.13, whose proof was inspired by their original proof.



**Theorem 4.2** ([4]). *Let  $P$  be a poset and  $J(P)$  the set of order ideals of  $P$ . If  $P$  is the disjoint union of two posets  $P = P_1 + P_2$ , then  $T(J(P)) = T(J(P_1)) \times T(J(P_2))$ . If  $P$  is not the disjoint union of two posets,  $T(J(P))$  is either the symmetric or alternating group on  $J(P)$ .*

*Proof.* If  $P$  is the disjoint union of two posets  $P = P_1 + P_2$ , then by Lemma 4.1, the toggle poset  $\mathcal{P}_{J(P)}$  is the direct product  $\mathcal{P}_{J(P_1)} \times \mathcal{P}_{J(P_2)}$ . So by Theorem 2.13,  $T(J(P)) = T(J(P_1)) \times T(J(P_2))$ .

We show that the family  $\{J(P) \mid P \text{ is a connected poset}\}$  is pleasant. If  $P$  is connected, then  $\mathcal{P}_{J(P)}$  is the partial order of  $J(P)$  by containment. Thus  $\mathcal{P}_{J(P)}$  has a single maximal element (the empty order ideal) and a single minimal element (the full order ideal), so  $\mathcal{P}_{J(P)}$  is not a toggle-disjoint direct sum. By Lemma 4.1,  $\mathcal{P}_{J(P)}$  is only a Cartesian product if  $P$  is disconnected. Thus condition (1) of Definition 2.12 is satisfied. Also, there will be either a maximal or minimal element  $e \in P$  such that  $P \setminus e$  is still connected, so this will suffice as the specified element in the definition of pleasant. (A maximal element will satisfy (3b) in Definition 2.12, while a minimal element will satisfy (3a).) Finally, we have checked using Sage [14] that for all connected posets with  $|P| \leq 5$ ,  $T(J(P)) = \mathfrak{S}_{J(P)}$  or  $A_{J(P)}$ , thus condition (2) of Definition 2.12 is satisfied. So this is a pleasant family, and the result follows by Theorem 2.13.  $\square$

In [4], P. Cameron and D. Fon-der-Flaass showed that the following action is expressible in terms of the toggle group.

**Definition 4.3.** Given a finite poset  $P$ , let the *rowmotion* of an order ideal  $I \in J(P)$  be defined as the order ideal generated by the minimal elements of  $P \setminus I$ .

**Theorem 4.4** ([4]). *Given any finite poset  $P$ , rowmotion acting on  $J(P)$  is equivalent to the toggle group element given by toggling the elements of  $P$  in the reverse order of any linear extension (that is, from top to bottom).*

Then in [17], J. Striker and N. Williams used the group structure of  $T(J(P))$  to prove the following.

**Theorem 4.5** ([17]). *If  $P$  is a ranked poset, given the proper notion of ‘left to right’, there is an equivariant bijection between  $J(P)$  under toggling from left to right (promotion) and  $J(P)$  under toggling from top to bottom (rowmotion).*

This theorem resulted in many interesting corollaries when applied to specific posets; see Sections 6 through 8 of [17].

**4.2. Chains.** Another set with combinatorially interesting toggle group is the set of chains  $\mathcal{C}(P)$  of a poset  $P$ . A *chain* in a poset is a set of mutually comparable elements. In this case and several cases to follow, but unlike in the case of order ideals, any element of a chain may always be toggled ‘out’ of the chain. That is, for any chain  $C$ , if  $p \in C$ , then  $t_p(C) = C \setminus \{p\}$ ; since  $C$  is a set of mutually comparable elements,  $C \setminus \{p\}$  is as well.

Recall from Lemma 4.1 that in the order ideal toggle group, two toggles commute if and only if there is not a covering relation between them. In contrast, we have the following toggle commutation relation for the chain toggle group.

**Lemma 4.6.** *Let  $P$  be a poset. In the chain toggle group  $T(\mathcal{C}(P))$ ,  $(t_p t_q)^2 = 1$  if and only if  $p$  and  $q$  are comparable in  $P$ .*

*Proof.* If  $p$  and  $q$  are incomparable in  $P$ , then  $t_p t_q(\emptyset) = \{q\}$  whereas  $t_q t_p(\emptyset) = \{p\}$ . So  $t_p$  and  $t_q$  do not commute. If  $p < q$ , then the presence or absence of  $p$  in a chain  $C$  does not affect whether  $q$  is in  $C$ , so  $t_p$  and  $t_q$  commute.  $\square$

We have the following structure theorem, as a corollary of Theorem 2.13.

**Corollary 4.7.** *Let  $P$  be a poset and  $\mathcal{C}(P)$  the set of chains of  $P$ . If  $P$  is the ordinal sum of two posets  $P = P_1 \oplus P_2$ , then  $T(\mathcal{C}(P)) = T(\mathcal{C}(P_1)) \times T(\mathcal{C}(P_2))$ . If  $P$  is not the ordinal sum of two posets,  $T(\mathcal{C}(P))$  is either the symmetric or alternating group on  $\mathcal{C}(P)$ .*

*Proof.* If  $P = P_1 \oplus P_2$ , then each element in  $P_1$  is less than every element in  $P_2$ . Thus, a chain  $\mathcal{C}$  in  $\mathcal{C}(P)$  consists of the union of any chain  $C_1 \in \mathcal{C}(P_1)$  and any chain  $C_2 \in \mathcal{C}(P_2)$ , where the choice of these chains is independent. By Lemma 4.6,  $t_{p_1}t_{p_2} = t_{p_2}t_{p_1}$  for any  $p_1 \in P_1, p_2 \in P_2$  since  $p_1 < p_2$  by the definition of ordinal sum. Thus the toggle poset  $\mathcal{P}_{\mathcal{C}(P)}$  will be the Cartesian product  $\mathcal{P}_{\mathcal{C}(P_1)} \times \mathcal{P}_{\mathcal{C}(P_2)}$ . Therefore, by Theorem 2.13,  $T(\mathcal{C}(P)) = T(\mathcal{C}(P_1)) \times T(\mathcal{C}(P_2))$ .

If  $P$  is not the ordinal sum of two posets, the result follows from Theorem 2.13 and the fact that the set  $\{\mathcal{C}(P) \mid P \text{ not an ordinal sum of two posets}\}$  is a pleasant family. For suppose  $P$  is not an ordinal sum. The toggle poset  $\mathcal{P}_{\mathcal{C}(P)}$  is the partial order of  $\mathcal{C}(P)$  by containment. Thus  $\mathcal{P}_{\mathcal{C}(P)}$  has a single minimal element (the empty chain), so  $\mathcal{P}_{\mathcal{C}(P)}$  is not a toggle-disjoint direct sum, and by Lemma 4.6,  $\mathcal{P}_{\mathcal{C}(P)}$  is only a Cartesian product if  $P$  is an ordinal sum. Thus condition (1) of Definition 2.12 is satisfied.

Also, if  $P$  is not an ordinal sum, then either  $P$  is an antichain, or there exists a minimal element  $x$  that is incomparable to some non-minimal element  $y$ . Suppose  $P$  is an antichain. Then any element may be chosen as the required  $e$  in condition (3b) of Definition 2.12.

Suppose  $P$  is not an antichain, so that there exists a minimal element  $x$  that is incomparable to some non-minimal element  $y$ . Pick any other minimal element  $e$  (another minimal element must exist, otherwise  $P$  would be the ordinal sum of  $x$  and the rest of the poset); this will be the required  $e$  in condition (3b) of Definition 2.12. This is because  $\mathcal{L}_{\bar{e}}$  for  $\mathcal{L} = \mathcal{C}(P)$  does not contain the subset  $\{x, y\}$ , since  $\{x, y\}$  is not a chain in  $P$ . Since  $P$  is not an ordinal sum and we are removing a minimal element  $e$ , the only spot where  $P \setminus e$  may now be an ordinal sum must involve the minimal elements, including  $x$ . But  $P \setminus e$  is not the ordinal sum of a subset involving  $x$  and the rest of the poset, since  $\{x, y\} \notin \mathcal{C}(P \setminus e)$ .

Finally, we have checked using Sage [14] that for all posets with  $|P| \leq 5$  such that  $P$  is not an ordinal sum,  $T(\mathcal{C}(P)) = \mathfrak{S}_{\mathcal{C}(P)}$  or  $A_{\mathcal{C}(P)}$ , thus condition (2) of Definition 2.12 is satisfied. So this is a pleasant family, and the result follows by Theorem 2.13.  $\square$

As in the case of order ideals (see Theorems 4.4 and 4.5 in Section 4.1), we have a theorem which says a large class of toggle orders are equivariant. Unlike in the case of order ideals, these toggle orders typically do not include toggles corresponding to all the elements of  $P$ . We will need the following lemma from [9], which appears as Lemma 5.1 in [17].

**Lemma 4.8.** *Let  $G$  be a group whose generators  $g_1, \dots, g_n$  satisfy  $g_i^2 = 1$  and  $(g_i g_j)^2 = 1$  if  $|i - j| > 1$ . Then for any  $\omega, \nu \in \mathfrak{S}_n$ ,  $\prod_i g_{\omega(i)}$  and  $\prod_i g_{\nu(i)}$  are conjugate.*

For  $C$  any chain of  $P$ , let  $t_C$  denote the composition of all the single-toggles  $t_p$  for  $p \in C$  (order does not matter, since by Lemma 4.6, the toggles within a chain commute). Also, for a collection of chains  $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$  and a permutation  $\pi \in \mathfrak{S}_k$ , let  $t_{\mathbf{C}(\pi)} := t_{C_{\pi(1)}} t_{C_{\pi(2)}} \cdots t_{C_{\pi(k)}}$ .

**Theorem 4.9.** *Fix a collection of chains  $\mathbf{C} = \{C_1, C_2, \dots, C_k\}$  in  $P$ . Suppose these chains have the property that each element of  $C_i$  is comparable with every element of  $C_j$  provided  $|i - j| > 1$ . Then for any  $\pi, \omega \in \mathfrak{S}_k$ , there is an equivariant bijection between  $\mathcal{C}(P)$  under  $t_{\mathbf{C}(\pi)}$  and under  $t_{\mathbf{C}(\omega)}$ .*

*Proof.* For any chain  $C \in P$ ,  $t_C^2 = 1$ , since all the toggles within a chain commute by Lemma 4.6. Then since each element of  $C_i$  is comparable with every element of  $C_j$  provided  $|i - j| > 1$ , this implies  $(t_{C_i} t_{C_j})^2 = 1$  for  $|i - j| > 1$ . Therefore, the result follows from Lemma 4.8 by considering the subgroup of  $T(\mathcal{C}(P))$  generated by the toggles  $\{t_p \mid p \in C_i \text{ for some } 1 \leq i \leq k\}$ .  $\square$

**4.3. Antichains.** An *antichain* in a poset  $P$  is a set of mutually incomparable elements; denote the set of antichains of  $P$  as  $\mathcal{A}(P)$ . Though antichains are in simple bijection with order ideals (the maximal elements of an order ideal are an antichain), antichain toggles differ from order ideal toggles. With antichains, as with chains, the elements may always be toggled ‘out’, whereas this is not the case in the order ideal toggle group.

**Lemma 4.10.** *Let  $P$  be a poset. In the antichain toggle group  $T(\mathcal{A}(P))$ ,  $(t_p t_q)^2 = 1$  if and only if  $p$  and  $q$  are incomparable in  $P$ .*

*Proof.* Note  $\emptyset$  and single element subsets are all in  $\mathcal{A}(P)$ . If  $p < q$  in  $P$ , then  $t_p t_q(\emptyset) = \{q\}$  whereas  $t_q t_p(\emptyset) = \{p\}$ . So  $t_p$  and  $t_q$  do not commute. If  $p$  and  $q$  are incomparable, then the presence or absence of  $p$  in an antichain  $A$  does not affect whether  $q$  is in  $A$ . So  $t_p$  and  $t_q$  commute.  $\square$

This differs from the case of order ideals, since  $t_p$  and  $t_q$  are noncommuting whenever  $p$  and  $q$  are comparable, not just when there is a cover between them.

But as in the case of order ideals, we have the following structure description as a corollary of Theorem 2.13.

**Corollary 4.11.** *Let  $P$  be a poset and  $\mathcal{A}(P)$  the set of antichains of  $P$ . If  $P$  is the disjoint union of two posets  $P = P_1 + P_2$ , then  $T(\mathcal{A}(P)) = T(\mathcal{A}(P_1)) \times T(\mathcal{A}(P_2))$ . If  $P$  is not the disjoint union of two posets,  $T(\mathcal{A}(P))$  is either the symmetric or alternating group on  $\mathcal{A}(P)$ .*

*Proof.* If  $P = P_1 + P_2$ , an antichain  $A$  in  $\mathcal{A}(P)$  consists of the union of any antichain  $A_1$  in  $\mathcal{A}(P_1)$  and any antichain  $A_2$  in  $\mathcal{A}(P_2)$ , where the choice of  $A_1$  and  $A_2$  is independent. Now since  $p_1$  and  $p_2$  are incomparable for every  $p_1 \in P_1$ ,  $p_2 \in P_2$ , we have by Lemma 4.10  $t_{p_1} t_{p_2} = t_{p_2} t_{p_1}$ . Thus, the toggle poset  $\mathcal{P}_{\mathcal{A}(P)}$  will be the Cartesian product  $\mathcal{P}_{\mathcal{A}(P_1)} \times \mathcal{P}_{\mathcal{A}(P_2)}$ . Therefore, by Theorem 2.13,  $T(\mathcal{A}(P)) = T(\mathcal{A}(P_1)) \times T(\mathcal{A}(P_2))$ .

If  $P$  is not the disjoint union of two posets, the result follows from Theorem 2.13 and the fact that the set  $\{\mathcal{A}(P) \mid P \text{ is a connected poset}\}$  is a pleasant family.

For suppose  $P$  is connected. The toggle poset  $\mathcal{P}_{\mathcal{A}(P)}$  is the partial order of  $\mathcal{A}(P)$  by containment. Thus  $\mathcal{P}_{\mathcal{A}(P)}$  has a single minimal element (the empty antichain), so  $\mathcal{P}_{\mathcal{A}(P)}$  is not a toggle-disjoint direct sum, and by Lemma 4.10,  $\mathcal{P}_{\mathcal{A}(P)}$  is only a Cartesian product if  $P$  is disconnected. Thus condition (1) of Definition 2.12 is satisfied.

Also, for given a connected poset  $P$ , you may choose any maximal or minimal element  $e \in P$  such that the poset  $P \setminus \{e\}$  is also connected to be the required  $e$  in condition (3b) of Definition 2.12.

Finally, we have checked using Sage [14] that for all connected posets with  $|P| \leq 5$ ,  $T(\mathcal{A}(P)) = \mathfrak{S}_{\mathcal{A}(P)}$  or  $A_{\mathcal{A}(P)}$ , thus condition (2) of Definition 2.12 is satisfied. So this is a pleasant family, and the result follows by Theorem 2.13.  $\square$

As in the cases of order ideals and chains, we have a theorem which says a large class of toggle orders are equivariant. Unlike in the case of order ideals, but as in the case of chains, these toggle orders typically do not include toggles corresponding to all the elements of  $P$ .

For  $A$  any antichain of  $P$ , let  $t_A$  denote the composition of all the single toggles  $t_p$  for  $p \in A$  (order does not matter, since by Lemma 4.10, the toggles within an antichain commute). Also, for a collection of antichains  $\mathbf{A} = \{A_1, A_2, \dots, A_k\}$  and a permutation  $\pi \in \mathfrak{S}_k$ , let  $t_{\mathbf{A}(\pi)} := t_{A_{\pi(1)}} t_{A_{\pi(2)}} \cdots t_{A_{\pi(k)}}$ .

**Theorem 4.12.** *Fix a collection of antichains  $\mathbf{A} = \{A_1, A_2, \dots, A_k\}$  in  $P$ . Suppose these antichains have the property that each element of  $A_i$  is incomparable with every element of  $A_j$  provided  $|i - j| > 1$ . Then for any  $\pi, \omega \in \mathfrak{S}_k$ , there is an equivariant bijection between  $\mathcal{A}(P)$  under  $t_{\mathbf{A}(\pi)}$  and under  $t_{\mathbf{A}(\omega)}$ .*

*Proof.* For any antichain  $A \in P$ ,  $t_A^2 = 1$ , since all the toggles within an antichain commute by Lemma 4.10. Then since each element of  $A_i$  is incomparable with every element of  $A_j$  provided  $|i - j| > 1$ , this implies  $(t_{A_i} t_{A_j})^2 = 1$  for  $|i - j| > 1$ . Therefore, the result follows from Lemma 4.8 by considering the subgroup of  $T(\mathcal{A}(P))$  generated by the toggles  $\{t_p \mid p \in A_i \text{ for some } 1 \leq i \leq k\}$ .  $\square$

**4.4. Interval-closed sets.** An *interval-closed set* in a poset  $P$  is a subset  $I \subseteq P$  such that if  $x, y \in I$  and  $x \leq y$ , then for any  $z \in P$  satisfying  $x \leq z \leq y$  we also have  $z \in I$ . Denote the set of interval-closed sets of a poset as  $\mathcal{IC}(P)$ .

Though the toggle group  $T(\mathcal{IC}(P))$  of interval-closed sets behaves similarly to the order ideal toggle group  $T(J(P))$ , we will see in Section 5 that *cover-closure* on  $\mathcal{IC}(P)$  acts quite differently than in the case of order ideals. In particular, we will not generally be able to express cover-closure as a product of toggles as in the case of rowmotion on order ideals.

We have the following toggle commutativity relation.

**Lemma 4.13.** *In the interval-closed set toggle group  $T(\mathcal{IC}(P))$ ,  $(t_p t_q)^2 = 1$  if and only if  $p$  and  $q$  are incomparable in  $P$  or (in the case  $p < q$ ) if  $q$  covers  $p$  where  $q$  is maximal and  $p$  is minimal.*

*Proof.* Note  $\emptyset \in \mathcal{IC}(P)$ . If  $p < q$  in  $P$  and  $q$  does not cover  $p$ , then  $t_p t_q(\emptyset) = \{q\}$  whereas  $t_q t_p(\emptyset) = \{p\}$ . So  $t_p$  and  $t_q$  do not commute.

If  $q$  covers  $p$  and there exists an element  $r$  covered by  $p$ , then  $t_p t_q(\{r\}) = \{r\}$  whereas  $t_q t_p(\{r\}) = \{p, q, r\}$ . If  $q$  covers  $p$  and there exists an element  $s$  covering  $q$ ,  $t_p t_q(\{s\}) = \{p, q, s\}$  whereas  $t_q t_p(\{s\}) = \{s\}$ . If no such  $r$  or  $s$  exists, that is, if  $p$  is minimal and  $q$  is maximal, then  $t_p$  and  $t_q$  commute. This is because the union of any interval-closed set with any subset of  $\{p, q\}$  is still an interval-closed set, so  $t_p$  and  $t_q$  always act nontrivially on any interval-closed set.

If  $p$  and  $q$  are incomparable, then by definition the presence or absence of  $p$  in an interval-closed set  $I$  does not affect whether  $q$  is in  $I$ . So  $t_p$  and  $t_q$  commute.  $\square$

We have the following structure description in Corollary 4.16 as a corollary of Theorem 2.13 and Lemma 4.13. First, we will need the following non-standard definitions.

**Definition 4.14.** Let  $P$  be a poset. If an element  $m \in P$  is a maximal element that only covers minimal elements or a minimal element that is only covered by maximal elements, call  $m$  *extremal-atomic*.

**Definition 4.15.** Given a poset  $P$  and a ordered subset of elements  $\{p_1, p_2, \dots, p_n\} \subset P$ , define  $P_i$  recursively as the induced subposet  $P_i := P_{i-1} \setminus p_i$  with  $P_0 := P$ . If  $P_n$  is a chain with at least three elements and for each  $0 \leq i \leq n - 1$ ,  $p_{i+1}$  is maximal or minimal in  $P_i$  and the poset  $P_i$  has connected Hasse diagram and no extremal-atomic element, call  $P$  *strongly-extremal-atomic-free*.

**Corollary 4.16.** *If  $P$  is the disjoint union of two posets  $P = P_1 + P_2$ , then  $T(\mathcal{IC}(P)) = T(\mathcal{IC}(P_1)) \times T(\mathcal{IC}(P_2))$ . Suppose  $P$  is not the disjoint union of two posets. If  $P$  contains an extremal-atomic element  $m$ , then  $T(\mathcal{IC}(P)) = T(\mathcal{IC}(P \setminus \{m\})) \times T(\mathcal{IC}(\{m\})) = T(\mathcal{IC}(P \setminus \{m\})) \times \mathfrak{S}_2$ . (So, in particular, if  $P$  has only two ranks, then  $T(P) = (\mathfrak{S}_2)^{|P|}$ .) If  $P$  is strongly-extremal-atomic-free, then  $T(\mathcal{IC}(P))$  is either the symmetric group or the alternating group on  $\mathcal{IC}(P)$ .*

*Proof.* If  $P = P_1 + P_2$ , an interval-closed set  $I$  in  $\mathcal{IC}(P)$  consists of the union of any interval-closed set  $I_1$  in  $\mathcal{IC}(P_1)$  and any interval-closed set  $I_2$  in  $\mathcal{IC}(P_2)$ , where the choice of  $I_1$  and  $I_2$  is independent. Now since  $p_1$  and  $p_2$  are incomparable for every  $p_1 \in P_1$ ,  $p_2 \in P_2$ , we have by Lemma 4.13,  $t_{p_1} t_{p_2} = t_{p_2} t_{p_1}$  so that the toggle poset is a Cartesian product. Therefore  $T(\mathcal{IC}(P)) = T(\mathcal{IC}(P_1)) \times T(\mathcal{IC}(P_2))$ .

If  $P$  contains an extremal-atomic element  $m$ , then by Lemma 4.13,  $m$  commutes with all other elements of  $P$ . So  $T(\mathcal{IC}(P)) = T(\mathcal{IC}(P \setminus \{m\})) \times T(\mathcal{IC}(\{m\})) = T(\mathcal{IC}(P \setminus \{m\})) \times \mathfrak{S}_2$ .

If  $P$  is strongly-extremal-atomic-free, the result follows from Theorem 2.13 if we can show that the set of interval-closed sets of strongly-extremal-atomic-free posets is a pleasant family. Given a strongly-extremal-atomic-free poset with its specialized sequence of elements  $\{p_1, p_2, \dots, p_n\}$ ,  $p_1$  will be the required  $e$  in (3b) of Definition 2.12. We have checked, using Sage [14], that for all strongly-extremal-atomic-free posets with  $|P| \leq 5$ ,  $T(\mathcal{IC}(P)) = \mathfrak{S}_{\mathcal{IC}(P)}$  or  $A_{\mathcal{IC}(P)}$ , thus (2) in Definition 2.12 is satisfied. And (1) is satisfied by Definition 4.15. So this is a pleasant family, and the result follows by Theorem 2.13.  $\square$

*Remark 4.17.* We speculate that in Corollary 4.16, the condition ‘strongly-extremal-atomic-free’ may be relaxed to ‘contains no extremal-atomic element’; we have checked, using Sage [14], this holds for  $|P| \leq 5$ . A proof in the case of this relaxed condition would then complete the toggle group structure theorem for interval-closed sets on any poset. For example, the poset  $P$  defined by the covers:  $a < b < c, a < d < e$  is not strongly-extremal-atomic-free. And yet, we have computed that  $|\mathcal{IC}(P)| = 25$  and  $T(\mathcal{IC}(P)) = \mathfrak{S}_{25}$ . Proving the interval-closed set toggle group structure description for all posets satisfying this relaxed condition would take some work beyond the statement of Theorem 2.13; note that no element of our example poset  $P$  above satisfies (3) in Definition 2.12 (and there is an infinite family of such posets).

**4.5. More than one partial order on the same set of elements.** Fix two partial orders  $\leq$  and  $\leq'$  on the same set  $P$ . Consider the toggle group  $T(\mathcal{L})$  for  $\mathcal{L} = J(P, \leq) \cup J(P, \leq')$ , the union of the two sets of order ideals. For  $p \in P$  and  $I \in \mathcal{L}$ , then, the toggle  $t_p$  is the symmetric difference of  $p$  and  $I$ , provided this is an order ideal in *either* of the two partial orders.

Given that the toggle group is often the full symmetric group, this could be a method for finding a bijection from one set of order ideals to another with the same set of elements.

We have the following (partial) commutation lemma, whose proof is immediate.

**Lemma 4.18.** *Given two partial orders  $\leq$  and  $\leq'$  on the same ground set  $P$ , consider the toggle group on the union of their sets of order ideals  $T(J(P, \leq) \cup J(P, \leq'))$ . In this toggle group, we have  $(t_p t_q)^2 = 1$  if  $p$  and  $q$  are incomparable in both  $(P, \leq)$  and  $(P, \leq')$ .*

This is only one among many toggle groups that could be constructed by combining various poset-theoretic structures. For example, it may be interesting to study the toggle group  $T(\mathcal{L})$  where  $\mathcal{L}$  is the union of order ideals and filters, or the union of order ideals and antichains, or the union of antichains under two different partial orders, etc.

**4.6. Independent sets of a graph.** We now move from the realm of posets to that of graphs. An *independent set* in a graph is a subset of the vertices such that no two have an edge between them. Let  $\mathcal{IS}(G)$  equal the set of independent sets of a graph  $G = (V(G), E(G))$ , where  $V(G)$  and  $E(G)$  are the vertex and edge sets, respectively. Since independent sets are subsets of the vertices, we take  $V(G)$  as our ground set and  $\mathcal{L} = \mathcal{IS}(G) \subseteq 2^{V(G)}$ .

With independent sets, as with chains and antichains, the elements may always be toggled ‘out’ of an independent set. We have the following toggle relation.

**Lemma 4.19.** *In the independent set toggle group  $T(\mathcal{IS}(G))$ ,  $(t_u t_v)^2 = 1$  if and only if there is no edge between  $u$  and  $v$  in  $G$ .*

*Proof.* If there is an edge between  $u$  and  $v$  in  $G$ , then  $t_u t_v(\emptyset) = \{v\}$  whereas  $t_v t_u(\emptyset) = \{u\}$ . So  $t_u$  and  $t_v$  do not commute. If there is no edge between  $u$  and  $v$ , then the presence or absence of  $u$  in an independent set  $I$  does not affect whether  $v$  is in  $I$ . So  $t_u$  and  $t_v$  commute.  $\square$

We have the following structure description as a corollary of Theorem 2.13 and Lemma 4.19.

**Corollary 4.20.** *Let  $G$  be a graph and  $\mathcal{IS}(G)$  the set of independent sets of  $G$ . If  $G$  is connected,  $T(\mathcal{IS}(G))$  is either the symmetric group or the alternating group on  $\mathcal{IS}(G)$ . If  $G$  has components  $G_1, G_2, \dots, G_k$ , then  $T(\mathcal{IS}(G)) = T(\mathcal{IS}(G_1)) \times T(\mathcal{IS}(G_2)) \times \dots \times T(\mathcal{IS}(G_k))$ .*

*Proof.* If  $G$  is disconnected with components  $G_1, G_2, \dots, G_k$ , an independent set  $X$  in  $\mathcal{IS}(G)$  consists of the disjoint union of the independent sets  $X_i$  in  $\mathcal{IS}(G_i)$ ,  $1 \leq i \leq k$ , where the choice of the  $X_i$  is independent. Thus the toggle poset  $\mathcal{P}_{\mathcal{IS}(G)}$  is the Cartesian product  $\mathcal{P}_{\mathcal{IS}(G_1)} \times \mathcal{P}_{\mathcal{IS}(G_2)} \times \dots \times \mathcal{P}_{\mathcal{IS}(G_k)}$ . Therefore, by Theorem 2.13,  $T(\mathcal{IS}(G)) = T(\mathcal{IS}(G_1)) \times T(\mathcal{IS}(G_2)) \times \dots \times T(\mathcal{IS}(G_k))$ .

If  $G$  is a connected graph, the result follows from Theorem 2.13 and the fact that the set  $\{\mathcal{IS}(G) \mid G \text{ is a connected graph}\}$  is a pleasant family.

For suppose  $G$  is connected. The toggle poset  $\mathcal{P}_{\mathcal{IS}(G)}$  is the partial order of  $\mathcal{IS}(G)$  by containment. Thus  $\mathcal{P}_{\mathcal{IS}(G)}$  has a single minimal element (the empty independent set), so  $\mathcal{P}_{\mathcal{IS}(G)}$  is not a toggle-disjoint direct sum, and by Lemma 4.10,  $\mathcal{P}_{\mathcal{IS}(G)}$  is only a Cartesian product if  $G$  is disconnected. Thus condition (1) of Definition 2.12 is satisfied.

Also, given a connected graph, you may choose any vertex that does not disconnect the graph when deleted to be the required  $e$  in condition (3b) of Definition 2.12.

Finally, we have checked using Sage [14] that for all connected graphs with  $|G| \leq 5$ ,  $T(\mathcal{IS}(P)) = \mathfrak{S}_{\mathcal{IS}(P)}$  or  $A_{\mathcal{IS}(P)}$ , thus condition (2) of Definition 2.12 is satisfied. So this is a pleasant family, and the result follows by Theorem 2.13.  $\square$

Toggling independent sets comes up in Monte-Carlo simulations and Glauber dynamics of the hard-core model of statistical physics.

Also, recently, D. Einstein, M. Farber, E. Gunawan, M. Joseph, M. Macauley, J. Propp, and S. Rubinstein-Salzedo have used this perspective of generalized toggling to prove some results on toggling noncrossing matchings and, more generally, independent sets of certain graphs [6].

**4.7. Vertex covers of a graph.** A *vertex cover* of a graph  $G$  is a subset of the vertices  $V(G)$  such that every edge in  $E(G)$  is incident to at least one vertex in the vertex cover. Denote the set of vertex covers of  $G$  as  $\mathcal{VC}(G)$ . Since vertex covers are subsets of the vertices, we take  $V(G)$  as our ground set and  $\mathcal{L} = \mathcal{VC}(G) \subseteq 2^{V(G)}$ . In the vertex cover toggle group  $T(\mathcal{VC}(G))$ , you may always toggle a vertex ‘in’ to a vertex cover, but you may not always toggle a vertex ‘out’.

We have the following toggle commutation lemma.

**Lemma 4.21.** *Let  $u$  and  $v$  be vertices of  $G$ . In the vertex cover toggle group  $T(\mathcal{VC}(G))$ ,  $(t_u t_v)^2 = 1$  if and only if there is no edge between  $u$  and  $v$ .*

*Proof.* Suppose there is an edge between  $u$  and  $v$ . Then consider the vertex cover  $X$  consisting of all the vertices in  $V(G)$ . We have that  $t_u t_v(X) = X \setminus \{v\}$  while  $t_v t_u(X) = X \setminus \{u\}$ . So  $t_u$  and  $t_v$  do not commute.

If  $u$  and  $v$  do not have an edge between them, then the presence or absence of  $u$  in a vertex cover has no affect on whether  $v$  must be in the vertex cover. So  $t_u$  and  $t_v$  commute.  $\square$

We have the following structure description. The proof is similar to the proof of Corollary 4.20, since the commutativity lemma is the same; the main difference is that condition (3a) of Definition 2.12 is used rather than condition (3b). Note we have checked the base cases  $|G| \leq 5$  using Sage [14].

**Corollary 4.22.** *Let  $G$  be a graph and  $\mathcal{VC}(G)$  the set of vertex covers of  $G$ . If  $G$  is connected,  $T(\mathcal{VC}(G))$  is either the symmetric group or the alternating group on  $\mathcal{VC}(G)$ . If  $G$  has components  $G_1, G_2, \dots, G_k$ , then  $T(\mathcal{VC}(G)) = T(\mathcal{VC}(G_1)) \times T(\mathcal{VC}(G_2)) \times \dots \times T(\mathcal{VC}(G_k))$ .*

Vertex covers have been used in commutative algebra work of B. Kubik, C. Paulsen, and S. Sather-Wagstaff to index the decomposition of edge ideals [11] and path ideals [10]. In this work, *minimal* vertex covers are of special importance. We can characterize a minimal vertex cover  $X$  as a vertex cover which is invariant under any of the toggles  $t_v$  for  $v \in X$ . It would be interesting to see what algebraic implications the action of the toggle group on vertex covers may have.

**4.8. Acyclic subgraphs of a graph.** In the next two subsections, we define other graph-theoretic toggle groups which we believe will be fruitful avenues for future study.

We consider an *acyclic subgraph* of a graph  $G$  to be a collection of edges in  $E(G)$  in which there is no cycle. Denote the set of acyclic subgraphs of  $G$  as  $\mathcal{AS}(G)$ . Since acyclic subgraphs are subsets of the edges rather than the vertices, we take  $E(G)$  as our ground set and  $\mathcal{L} = \mathcal{AS}(G) \subseteq 2^{E(G)}$ .

The maximal acyclic subgraphs are the spanning forests. In the acyclic subgraph toggle group  $T(\mathcal{AS}(G))$ , you may always toggle an edge ‘out’, but you may not always toggle an edge ‘in’, since adding an edge may introduce a cycle.

We have the following commutation lemma, but leave a toggle group structure description to future work.

**Lemma 4.23.** *Let  $e$  and  $f$  be edges of  $G$ . In the acyclic subgraph toggle group  $T(\mathcal{AS}(G))$ ,  $(t_e t_f)^2 = 1$  if and only if no cycle of  $G$  contains both  $e$  and  $f$ .*

*Proof.* Suppose a cycle  $C$  of  $G$  contains edges  $e$  and  $f$ ; consider the acyclic subgraph  $C \setminus e$ . We know  $t_e t_f(C \setminus e) = C \setminus f$  since both toggles act nontrivially whereas  $t_f t_e(C \setminus e) = C \setminus \{e, f\}$  since here  $t_e$  must act as the identity (or else the resulting subgraph would be a cycle). So  $t_e$  and  $t_f$  do not commute.

If no cycle contains both  $e$  and  $f$ , then the presence or absence of  $e$  in a subgraph does not affect whether  $f$  may be in the subgraph while maintaining acyclicity, and vice versa. So in this case  $t_e$  and  $t_f$  commute.  $\square$

**4.9. Connected subgraphs of a graph.** We consider a *connected subgraph* of a graph  $G$  to be a collection of edges in  $E(G)$ , which together with the vertices that are the endpoints of these edges, forms a connected graph. (We consider the empty subgraph to be a connected subgraph, but a graph consisting of a single vertex does not satisfy our definition of connected subgraph.) So we take our ground set to be the set of edges  $E(G)$ , and our subsets  $\mathcal{L} \subseteq 2^{E(G)}$  to be the edge sets of connected subgraphs. So we define the connected subgraph toggle group as the subgroup of the symmetric group on all connected subgraphs generated by the edge toggles.

It is not true that you may always toggle an edge ‘in’, since that may disconnect the graph if the edge is not connected to the subgraph on which the toggle is acting. And it is not true that you may always toggle an edge ‘out’, since that may also disconnect the graph in the case the edge is a *bridge* of the subgraph. In this sense, this toggle group is similar to the case of order ideals in a poset, but unlike in the case of order ideals, connected subgraphs are closed under neither union or intersection.

It may be interesting to investigate the subset-toggle group in which you allow toggling by all connected subgraphs, or some subset of connected subgraphs. For example, it may be interesting to toggle by paths, cycles, or paths/cycles of a certain size. We leave as an interesting avenue for further research the investigation of the single- and subset-toggle groups on connected subgraphs.

**4.10. Matroids.** A matroid is an abstraction of the notion of linear independence. Matroids can be defined in many ways; we give two definitions here and give characterizations of these definitions in terms of toggles.

**Definition 4.24.** A finite *matroid*  $M$  may be defined as a pair  $(E, \mathcal{I})$ , where  $E$  is a finite set (called the ground set) and  $\mathcal{I}$  is a set of subsets of  $E$  (called the independent sets) with the following properties:

- (1)  $\emptyset \in \mathcal{I}$ ;
- (2) For each  $X \subseteq Y \subseteq E$ , if  $Y \in \mathcal{I}$  then  $X \in \mathcal{I}$  (the *hereditary* property); and
- (3) If  $X, Y \in \mathcal{I}$  and  $|Y| > |X|$ , then there exists  $y \in Y \setminus X$  such that  $X \cup \{y\} \in \mathcal{I}$  (the *independent set exchange* property).

Since the set  $\mathcal{I}$  of independent sets is a set of subsets of  $E$ , we could define and study the toggle group  $T(\mathcal{I})$ . For any  $X \in \mathcal{I}$  and any  $x \in X$ ,  $t_x(X) = X \setminus \{x\}$ . That is, we may always toggle elements ‘out’ of an independent set in the toggle group of independent sets of a matroid.

*Remark 4.25.* We can characterize the independent set exchange condition above using the toggle group  $T(\mathcal{I})$ . Namely, (3) is equivalent to the following:

- (3\*) If  $X, Y \in \mathcal{I}$  and  $|Y| > |X|$ , then there exists  $y \in Y \setminus X$  such that  $t_y(X) \neq X$  (that is,  $t_y$  acts nontrivially on  $X$  in  $T(\mathcal{I})$ ).

The maximal independent sets of a matroid are called the bases; a matroid may be defined in terms of the bases rather than the independent sets.

**Definition 4.26.** A finite *matroid*  $M$  may be defined as a pair  $(E, \mathcal{B})$ , where  $E$  is a finite set (called the ground set) and  $\mathcal{B}$  is a collection of subsets of  $E$  (called the bases), with the following properties:

- (1)  $\mathcal{B}$  is nonempty; and
- (2) If  $X, Y \in \mathcal{B}$  and  $x \in X \setminus Y$ , then there exists  $y \in Y \setminus X$  such that  $(X \setminus \{x\}) \cup \{y\} \in \mathcal{B}$  (the *basis exchange condition*).

Since the set of bases  $\mathcal{B}$  is a set of subsets of  $E$ , we could define and study a toggle group  $T(\mathcal{B})$ . But the single-toggle group on the bases will be uninteresting; single-toggles will all act as the identity, since the bases are the *maximal* independent sets. Therefore, a more interesting toggle group on the bases would be the power set toggle group or a subset-toggle group; see Section 3 for relevant definitions.

*Remark 4.27.* We can characterize the basis exchange condition above using the power set toggle group on the bases. Namely, (2) is equivalent to the following:

- (2\*) If  $X, Y \in \mathcal{B}$  and  $x \in X \setminus Y$ , then there exists  $y \in Y \setminus X$  such that in the power set toggle group  $T_{2^E}(\mathcal{B})$ , the subset-toggle  $t_{\{x,y\}}(X) \neq X$  (that is,  $t_{\{x,y\}}$  acts nontrivially on  $X$  in  $T_{2^E}(\mathcal{B})$ ).

**4.11. Convex geometries.** Just as matroids are an abstraction of the notion of linear independence, convex geometries are an abstraction of the notion of convexity. (Convex geometries are dual to *antimatroids* and also equivalent to *meet-distributive lattices*.) As with matroids, we first state the definition and then characterize part of the definition in terms of toggles.

**Definition 4.28.** Let  $E$  be a finite nonempty set. A set  $\mathcal{L}$  of subsets of  $E$  is a *convex geometry* on  $E$  if  $\mathcal{L}$  satisfies the following properties.

- (1)  $\emptyset \in \mathcal{L}$  and  $E \in \mathcal{L}$ ;
- (2) If  $X, Y \in \mathcal{L}$ , then  $X \cap Y \in \mathcal{L}$ ; and
- (3) If  $X \in \mathcal{L} \setminus \{E\}$ , then there exists  $e \in E \setminus X$  such that  $X \cup \{e\} \in \mathcal{L}$ .

The elements of  $\mathcal{L}$  are called the *convex sets*.

*Remark 4.29.* We can recast item (3) above in terms of toggles. Namely,

- (3\*) For any  $X \in \mathcal{L} \setminus \{E\}$ , there exists a toggle  $t_e$  with  $e \in E \setminus X$  such that  $t_e(X) \neq X$ .

**Example 4.30.** Let  $E = \{1, 2, 3\}$  and  $\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$ .  $\mathcal{L}$  is a convex geometry, but note that  $\mathcal{L}$  is not closed under unions since  $\{1, 3\} \notin \mathcal{L}$ . So, in particular, there is no poset  $P$  such that  $\mathcal{L} = J(P)$ .

We now relax our definition to allow  $E$  to be infinite and also to allow that  $\mathcal{L}$  not include  $E$ . So we define an *extended convex geometry* by replacing conditions (1) and (3) as follows.



**Definition 4.31.** Let  $E$  be a nonempty set. A family  $\mathcal{L}$  of subsets of  $E$  is an *extended convex geometry* on  $E$  if  $\mathcal{L}$  satisfies the following properties.

- (1')  $\emptyset \in \mathcal{L}$
- (2) If  $X, Y \in \mathcal{L}$ , then  $X \cap Y \in \mathcal{L}$ .
- (3') If  $X$  is not a maximal element in  $\mathcal{L}$ , then there exists  $e \in E \setminus X$  such that  $X \cup \{e\} \in \mathcal{L}$ .

Note that a convex geometry is also an extended convex geometry.

Convex geometries have a closure operation, defined below; this is also well-defined for extended convex geometries, since it only relies on condition (2) which remains unchanged.

**Definition 4.32.** For  $A \subseteq E$  and  $\mathcal{L} \subseteq 2^E$  an extended convex geometry, define the *closure operator* of  $\mathcal{L}$ ,  $\tau_{\mathcal{L}} : 2^E \rightarrow \mathcal{L}$  as

$$\tau_{\mathcal{L}}(A) = \bigcap \{X \in \mathcal{L} \mid A \subseteq X\}.$$

We may define *cover-closure* for extended convex geometries using this closure operation, see Section 5.

Several of the previous examples are also examples of convex geometries or extended convex geometries. In particular, we show below that the set of interval-closed sets of a poset  $\mathcal{IC}(P)$  (Section 4.4) is a convex geometry and the set of chains of a poset  $\mathcal{C}(P)$  (Section 4.2) is an extended convex geometry.

**Proposition 4.33.**  $\mathcal{IC}(P)$  is a convex geometry.

*Proof.* The empty set and  $P$  are both convex sets, so condition (1) is satisfied. The intersection of any two interval-closed sets is also interval-closed, satisfying condition (2). Condition (3) is satisfied since, given  $I \in \mathcal{IC}(P)$ ,  $I \neq P$ , and  $x \in P$ , if  $I \cup \{x\} \notin \mathcal{IC}(P)$ , then there must be  $y_1 \in P$  such that the interval-closed set induced by  $I \cup \{x\}$  contains  $y_1$ . But then either  $I \cup \{y_1\} \in \mathcal{IC}(P)$  or there exists  $y_2 \in P$  such that the interval-closed set induced by  $I \cup \{y_1\}$  contains  $y_2$ . Since the poset is finite and the partial order is acyclic, after a finite number of steps we will find  $y_i$  such that  $I \cup \{y_i\} \in \mathcal{IC}(P)$ .  $\square$

**Proposition 4.34.**  $\mathcal{C}(P)$  is an extended convex geometry.

*Proof.* The empty set is a chain, so condition (1') is satisfied. The set of chains is closed under intersection, so condition (2) is satisfied. To any non-maximal chain  $C$ , we can add any single element of a maximal chain containing  $C$  to  $C$  to create a chain of length one greater, so condition (3') is satisfied.  $\square$

The set of all convex geometries is not a pleasant family, so we cannot use Theorem 2.13 to obtain a structure theorem for all convex geometries. Consider the following example.

**Example 4.35.** Let  $E$  and  $\mathcal{L}$  be as in Example 4.30. This is a convex geometry, by Definition 4.28.

But note that the toggle posets  $\mathcal{P}_{\mathcal{L}_1} = \mathcal{P}_{\mathcal{L}_2} = \mathcal{P}_{\mathcal{L}_3} = \diamond$ , which is the Cartesian product of two chains. Thus  $T(\mathcal{L}_1) = T(\mathcal{L}_2) = T(\mathcal{L}_3) = \mathfrak{S}_3 \times \mathfrak{S}_3$ . Therefore, convex geometries are not a pleasant family, since there is no choice of  $e \in E$  that satisfies condition (3) of Definition 2.12.

In the next section, we define a convex closure action and show that under certain conditions, we can tell whether the space it is acting on is a convex geometry.

## 5. COVER-CLOSURE AND ROWMOTION

In this section, we define an action we call *cover-closure* on any set of subsets  $\mathcal{L} \subseteq 2^E$  ( $E$  countable) with a closure operator. We show in Lemma 5.5 that if  $\mathcal{L}$  is the set of order ideals of a poset, cover-closure is exactly the rowmotion action of [4, 17]. We prove in Theorem 5.6 that if

$E$  is finite and cover-closure is injective, then  $\mathcal{L}$  is a convex geometry. In Conjecture 5.8 we assert something stronger, namely, if  $E$  is finite and cover-closure is injective, then  $\mathcal{L}$  must be the set of order ideals  $J(P)$  for some poset  $P$ .

**Definition 5.1.** Given a ground set  $E$ , a *closure operator* is a function  $\tau : 2^E \rightarrow 2^E$  that satisfies the following for all  $X, Y \subseteq E$ :

- (1)  $X \in \tau(X)$  (*extensivity*);
- (2)  $X \subseteq Y$  implies  $\tau(X) \subseteq \tau(Y)$  (*monotonicity*); and
- (3)  $\tau(\tau(X)) = \tau(X)$  (*idempotence*).

A *closed set* with respect to  $\tau$  is a set  $X \subseteq E$  for which  $\tau(X) = X$ .

Using this closure operator, we define the following action.

**Definition 5.2.** Let  $E$  be a countable set with closure operator  $\tau$ , and fix  $\mathcal{L} \subseteq 2^E$ . For  $X \in 2^E$ , let  $\text{cov}(X) \subseteq E \setminus X$  be the maximal subset of  $E \setminus X$  such that  $\forall e \in \text{cov}(X), X \cup \{e\} \in \mathcal{L}$ . Call  $\text{cov}(X)$  the set of *covers* of  $X$ . Then we define *cover-closure*  $\xi : 2^E \rightarrow 2^E$  as  $\xi(X) = \tau(\text{cov}(X))$ .

*Remark 5.3.* Note that if  $X \in \mathcal{L}$ ,  $\text{cov}(X)$  is the set of covers of  $X$  in the toggle poset  $\mathcal{P}_{\mathcal{L}}$  of Definition 2.5.

**Example 5.4.** Let  $E = \{1, 2, 3, 4\}$  and

$$\mathcal{L} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}\}.$$

Then the action of cover-closure  $\xi$  on  $\mathcal{L}$  is given in Figure 4.

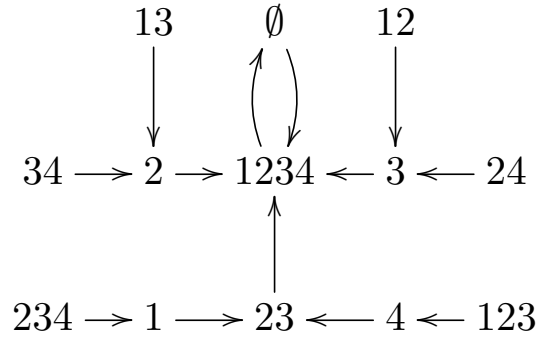


FIGURE 4. The action of cover-closure  $\xi$ , where  $\mathcal{L}$  is as in Example 5.4

The following lemma shows that cover-closure on the set of order ideals  $J(P)$  is the *rowmotion* action of [4, 17].

**Lemma 5.5.** *If  $\mathcal{L} = J(P)$  for some poset  $P$  and  $\tau(X)$  is the intersection of all the order ideals in  $J(P)$  containing  $X$ , then cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is rowmotion on  $J(P)$ .*

*Proof.* Rowmotion on an order ideal  $I$  is defined as the order ideal generated by the minimal elements of  $P \setminus I$ . (See Definition 4.3.) Thus if  $\mathcal{L} = J(P)$  and  $\tau(X)$  is the intersection of all order ideals containing  $X$ , we have that cover-closure  $\xi$  is exactly rowmotion.  $\square$

We now show that if cover-closure is bijective on  $\mathcal{L}$ ,  $\mathcal{L}$  must be a convex geometry.

**Theorem 5.6.** *Given a finite ground set  $E$  and a closure operator  $\tau$  with set of closed sets  $\mathcal{L} \subseteq 2^E$ , if cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is injective and thus bijective, then  $\mathcal{L}$  must be a convex geometry.*

*Proof.* Let  $E$  be a finite ground set with closure operator  $\tau$  and set of closed sets  $\mathcal{L} \subseteq 2^E$ . It is well-known that the set of closed sets is a lattice, where  $X \wedge Y = X \cap Y$  and  $X \vee Y = \tau(X \cup Y)$ . But in order to keep this proof self-contained, we prove here that if  $X, Y \in \mathcal{L}$  then  $X \cap Y \in \mathcal{L}$ . We need only show  $\tau(X \cap Y) = X \cap Y$ , since  $\mathcal{L}$  is the set of closed sets under  $\tau$ . Now  $X \cap Y \subseteq \tau(X \cap Y)$  by extensivity. But  $\tau(X \cap Y) \subseteq \tau(X)$  and  $\tau(X \cap Y) \subseteq \tau(Y)$  by monotonicity.  $\tau(X) = X$  and  $\tau(Y) = Y$  since  $X, Y \in \mathcal{L}$ . So  $\tau(X \cap Y) \subseteq X \cap Y$ . Thus  $\tau(X \cap Y) = X \cap Y$ . Therefore, we have shown condition (2) of Definition 4.28.

Suppose  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is injective and thus bijective. Now, since  $E$  is finite, there must be  $X_{max} \in \mathcal{L}$  such that  $|X_{max}| \geq |X|$  for any  $X \in \mathcal{L}$ . We know that  $\text{cov}(X_{max}) = \emptyset$ , so  $\xi(X_{max}) = \tau(\emptyset) = \emptyset$ . Thus  $\emptyset \in \mathcal{L}$  since  $\xi : \mathcal{L} \rightarrow \mathcal{L}$ . Also,  $X_{max}$  is the unique maximal element of  $\mathcal{L}$ , since if there were two distinct subsets in  $\mathcal{L}$  contained by no other subset of  $\mathcal{L}$ , there would be two subsets mapped to  $\emptyset$  by  $\xi$ . We assume  $X_{max} = E$ , otherwise, reduce the size of the ground set  $E$  by removing the elements that do not appear in any subset in  $\mathcal{L}$ . Thus condition (1) of Definition 4.28 is satisfied.

Now, cover-closure on  $X \in \mathcal{L}$  is defined as  $\xi(X) = \tau(\text{cov}(X))$  where  $\text{cov}(X)$  is the set of elements of  $E$  which can be singly added to  $X$ . That is,  $\text{cov}(X)$  is the set of edge labels for every cover of  $X$  in  $\mathcal{P}_{\mathcal{L}}$ . Since cover-closure is a bijection, there must be a one-to-one correspondence between  $X$  and  $\text{cov}(X)$  for all  $X \in \mathcal{L}$ . In particular, the set  $\text{cov}(X)$  is unique for each  $X$ , otherwise  $\xi$  would map two distinct elements of  $\mathcal{L}$  to the same image.

Now since cover-closure is a bijection, for each  $X \in \mathcal{L}$  such that  $X \neq E$ , there exists  $e \in E \setminus X$  such that  $X \cup \{e\} \in \mathcal{L}$ , otherwise  $\text{cov}(X) = \emptyset$  and  $\xi(X) = \emptyset = \xi(E)$  which would be a contradiction. So condition (3) of Definition 4.28 is satisfied, and  $\mathcal{L}$  is a convex geometry.  $\square$

We have found no examples of a convex geometry  $\mathcal{L}$  not equal to the set of order ideals of a poset in which cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is bijective. For example, cover-closure is not bijective in the following convex geometry.

**Example 5.7.** Let  $E$  and  $\mathcal{L}$  be as in Example 4.30. This is a convex geometry, but

$$\xi(\{2\}) = \tau(\{1, 3\}) = \{1, 2, 3\} = \xi(\emptyset),$$

so cover-closure is not bijective.

This discussion leads us to conjecture the following strengthening of Theorem 5.6.

**Conjecture 5.8.** *Given a finite ground set  $E$  and a closure operator  $\tau$  with set of closed sets  $\mathcal{L} \subseteq 2^E$ , if cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is injective and thus bijective, then  $\mathcal{L} = J(P)$  for some poset  $P$ .*

Together with Lemma 5.5, this conjecture would imply that cover-closure  $\xi : \mathcal{L} \rightarrow \mathcal{L}$  is bijective if and only if  $\mathcal{L} = J(P)$  for some poset  $P$ .

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